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THE FINITE FOURIER SERIES II. THE HARMONIC ANALYSIS OF SKEW POL--ETC(U)

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THE FINITE FOURIER SERIES II. THE HARMONIC ANALYSIS OF SKEW POLYONS AS A SOURCE OF OUTDOOR SCULPTURES

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ABSTRACT

In [1, 125] and again in [3] Jesse Douglas established the following

Theorem 1. Let

$$\Pi = (z_0, z_1, z_2, z_3, z_4), \qquad (z_{\nu+5} = z_\nu),$$

be a closed skew pentagon in R^3, viewed as a vector space. Let

$$z'_\nu = \frac{1}{2}(z_{\nu+2} + z_{\nu-2}) \qquad (\nu = 0,1,2,3,4)$$

be the midpoint of the side $[z_{\nu-2}, z_{\nu+2}]$ which is opposite to the vertex z_ν.

For each ν determine on the line joining z_ν to z'_ν, the points f^1_ν, f^2_ν, such that

$$f^1_\nu - z'_\nu = \frac{1}{\sqrt{5}}(z'_\nu - z_\nu), \quad f^2_\nu - z'_\nu = -\frac{1}{\sqrt{5}}(z'_\nu - z_\nu) .$$

Then

$$\Pi^1 = (f^1_0, f^1_1, f^1_2, f^1_3, f^1_4)$$

is a plane and affine regular pentagon, and

$$\Pi^2 = (f^2_0, f^2_1, f^2_2, f^2_3, f^2_4)$$

is a plane and affine regular starshaped pentagon.

By an affine regular (starshaped) pentagon we mean an affine image of a regular (starshaped) pentagon.

It is shown here that the natural and inevitable source of Theorem 1 is the finite Fourier series of five terms. The affine regular pentagons Π^1 and Π^2 represent essentially the harmonic analysis of the pentagon Π. Placing the origin O of R^3 in the centroid of the vertices of Π, the complete harmonic analysis of Π is summarized by the relation

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$$\Pi = \frac{1-\sqrt{5}}{2}\,\Pi^1 + \frac{1+\sqrt{5}}{2}\,\Pi^2 \ .$$

The Figure 1 shows a 2-dimensional illustration of Theorem 1, but this gives only a faint idea of the appearance of a 3-dimensional structure. The author made a 3-dimensional structure out of 20 thin wooden sticks, and was struck by its appropriateness as a source of outdoor sculptures.

Theorem 2 (§4) describes the analogue of Theorem 1 for skew heptagons in R^3. Figure 3, of §5, shows a 2-dimensional illustration of Theorem 2. A 3-dimensional model would be very desirable.

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Significance and Explanation

It is shown that a beautiful theorem of Jesse Douglas, of Plateau problem fame, on skew pentagons [3], should be derived by using the so-called finite Fourier series. Douglas' result (stated as Theorem 1 in our Introduction) states that in a certain figure formed of ten straight lines in space, there appear two pentagons Π^1 and Π^2 which are plane pentagons and affine regular, meaning the following: Π^1 looks like a regular pentagon which is viewed for some distance (slantingly) in space. Likewise Π^2 looks like a regular starshaped pentagon seen under similar circumstances. The entire figure depends on the arbitrary choice of a skew pentagon Π in space. Our Figure 1 shows the case when the pentagon Π, having the vertices z_0, z_1, z_2, z_3, z_4, is in a plane. This, however, gives only a faint idea of the aspect of a 3-dimensional structure. The author made a 3-dimensional structure out of 20 thin wooden sticks, and he was struck by its appropriateness as a source of outdoor sculptures. Theorem 2 (§4) describes the analogue of Theorem 1 for skew heptagons (7-sided polygons) in space. Figure 3, of §5, shows a 2-dimensional illustration of Theorem 2. The author is now making an illustration of Theorem 2 in space.

THE FINITE FOURIER SERIES II.
THE HARMONIC ANALYSIS OF SKEW POLYONS
AS A SOURCE OF OUTDOOR SCULPTURES

I. J. Schoenberg

1. <u>Introduction</u>. The previous paper [4] on the subject of the finite Fourier series (f.F.s) dealt with some known and some new applications to problems of elementary geometry. In the present second paper ~~we~~ are appl~~y~~ied ~~it~~ to a ~~beautiful~~ theorem of Jesse Douglas ~~[3]~~ [2] on skew pentagons in space. It is shown here that Douglas' theorem amounts to the graphical harmonic analysis of skew pentagons and that it is also the source of striking outdoor sculptures. This last opinion is shared by two great art experts, Allan and Marjorie McNab, whom I wish to thank for their encouragement.

The case of a pentagon is discussed in §§2 and 3. Again with possible sculptures in mind, we present in §§4 and 5 the harmonic analysis of a skew heptagon.

The theorem mentioned above is as follows. (See Figure 1).

<u>Theorem 1.</u> (J. Douglas). <u>Let</u>

(1.1) $$\Pi = (z_0, z_1, z_2, z_3, z_4), \qquad (z_{\nu+5} = z_\nu),$$

<u>be a skew closed pentagon in</u> R^3, <u>viewed as a vector space. Let</u>

(1.2) $$z'_\nu = \frac{1}{2}(z_{\nu+2} + z_{\nu-2}) \qquad (\nu = 0,1,2,3,4)$$

<u>be the midpoint of the side</u> $[z_{\nu-2}, z_{\nu+2}]$ <u>which is opposite to the vertex</u> z_ν.

<u>For each</u> ν <u>determine, on the line joining</u> z_ν <u>to</u> z'_ν, <u>the points</u> f^1_ν, f^2_ν <u>such that</u>

(1.3) $$f^1_\nu - z'_\nu = \frac{1}{\sqrt{5}}(z'_\nu - z_\nu), \quad f^2_\nu - z'_\nu = -\frac{1}{\sqrt{5}}(z'_\nu - z_\nu).$$

<u>then</u>

(1.4) $$\Pi^1 = (f^1_0, f^1_1, f^1_2, f^1_3, f^1_4)$$

<u>is a plane and affine regular pentagon, and</u>

(1.5) $$\Pi^2 = (f^2_0, f^2_1, f^2_2, f^2_3, f^2_4)$$

<u>is a plane and affine regular starshaped pentagon.</u>

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By an affine regular (starshaped) pentagon we mean an affine image of a regular (starshaped) pentagon.

Theorem 1 was easy to verify, but was not easily discovered. In several papers [1], [2], [3], Douglas thoroughly explores these problems. He uses the classical eigenvalue properties of cyclic (or circulant) square matrices. Theorem 1 is stated as an example of general results in [1, 125], and is also proved directly in [3], with a short ad hoc proof which does not seem to be particularly transparent. The author's contributions go in two different directions.

1. The natural foundation of Douglas' theory seems to be the finite Fourier series. To be sure, the f.F.s. is essentially equivalent to the properties of cyclic matrices used by Douglas. However, it is shown in §2 that if we invert the f.F.s. for a pentagon, not in its usual complex form, but in its so-called real form, we are inevitably led to Douglas' Theorem 1. From this point of view Douglas' idea easily generalizes to the harmonic analysis of skew heptagons in R^3 (Theorem 2 of §4).

2. The author constructed out of 20 thin wooden sticks a 3-dimensional model, well over two feet in size, illustrating Theorem 1. The appearance of the plane affine regular pentagons Π^1 and Π^2 was expected, but enjoyable just the same, especially as they lie in two different planes. For contrast, the sides of the pentagons Π, Π^1, Π^2, were painted in three different colors. The shape of the entire structure, i.e. ignoring rigid motions, depends on 9 real parameters. This diversity and total lack of symmetry allows for artistic effects and makes the presence of the affine regular pentagons more striking: Order out of chaos. Made of metal bars and of a more heroic size, it would provide a striking outdoor sculpture. Our Figure 1 shows the case when the pentagon Π, having the vertices z_0, z_1, z_2, z_3, z_4, is in a plane. This, however, gives only a faint idea of the aspect of a 3-dimensional structure.

We also construct a 3-dimensional illustration of Theorem 2 out of 63 thin wooden sticks. Based on a skew heptagon Π, it shows the three affine regular heptagons Π^1, Π^2, Π^3, painted in three contrasting colors. This model is yet to be shown to the art experts for their comments on its suitablilty as an outdoor sculpture. Our Figure 3 shows an example when the heptagon $\Pi = (z_0, z_1, \ldots, z_6)$ is in the plane.

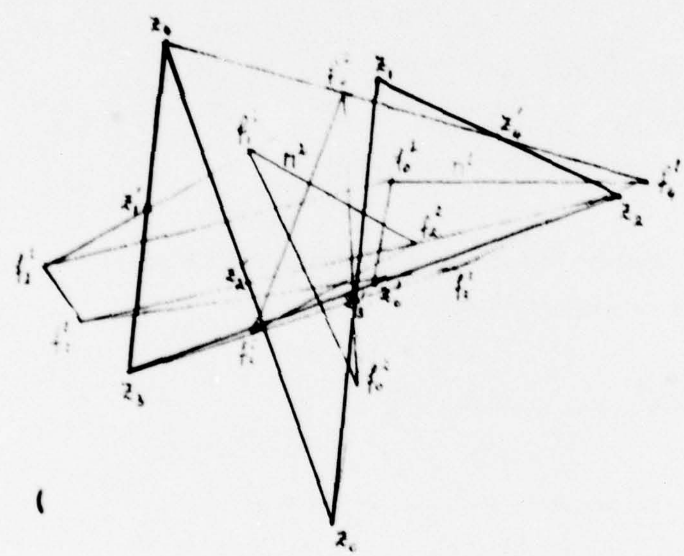

Figure 1

2. <u>A proof of Theorem 1 for pentagons Π in the complex plane.</u> If $\Pi \subset \mathbb{C}$ we can consider all symbols $z_\nu, z'_\nu, f^1_\nu, f^2_\nu$, of Theorem 1, as complex numbers. With $\omega_\nu = \exp(2\pi i\nu/5)$, the f.F.s. of the z_ν is the expansion

$$z_\nu = \zeta_0 + \zeta_1\omega_\nu + \zeta_2\omega_\nu^2 + \zeta_2\omega_\nu^3 + \zeta_4\omega_\nu^4 \qquad (\nu = 0,\ldots,4) \tag{2.1}$$

where the f.F. coefficients ζ_ν are given by the inverse formulae

$$\zeta_\nu = \frac{1}{5}(z_0 + z_1\omega_\nu^{-} + z_2\omega_\nu^{-2} + z_3\omega_\nu^{-3} + z_4\omega_\nu^{-4}) . \tag{2.2}$$

Both formulae extend the definitions of (z_ν) and (ζ_ν) to periodic sequences of period 5. Since $\zeta_3 = \zeta_{-2}$, $\zeta_4 = \zeta_{-1}$, we may rewrite (2.1) as

$$z_\nu = \zeta_0 + (\zeta_1\omega_\nu + \zeta_{-1}\omega_\nu^{-1}) + (\zeta_2\omega_\nu^2 + \zeta_{-2}\omega_\nu^{-2}) \tag{2.3}$$

which is the so-called <u>real</u> f.F.s. of the (z_ν). Writing

$$\tilde{f}^1_\nu = \zeta_1\omega_\nu + \zeta_{-1}\omega_\nu^{-1}, \quad \tilde{f}^2_\nu = \zeta_2\omega_\nu^2 + \zeta_{-2}\omega_\nu^{-2}, \tag{2.4}$$

we obtain the final form of the f.F.s. as

$$z_\nu = \zeta_0 + \tilde{f}^1_\nu + \tilde{f}^2_\nu . \tag{2.5}$$

By (2.2) ζ_0 is the centroid of the z_ν. Selecting this centroid as the origin O of the complex plane, (2.5) simplifies to

$$z_\nu = \tilde{f}^1_\nu + \tilde{f}^2_\nu \qquad (\nu = 0,\ldots,4). \tag{2.6}$$

Introducing the two new pentagons

$$\tilde{\Pi}^1 = (\tilde{f}^1_\nu) \quad \text{and} \quad \tilde{\Pi}^2 = (\tilde{f}^2_\nu) \tag{2.7}$$

we may represent the pentagon $\Pi = (z_\nu)$ in the form

$$\Pi = \tilde{\Pi}^1 + \tilde{\Pi}^2 . \tag{2.8}$$

The simple nature of the pentagons (2.7) is shown by the following statements:

(2.9) $\tilde{\Pi}^1$ <u>is an affine regular pentagon,</u>

(2.10) $\tilde{\Pi}^2$ <u>is an affine regular starshaped pentagon.</u>

A proof is immediate: Setting in the first relation (2.4) $\tilde{f}^1_\nu = x_\nu + iy_\nu$, $\zeta_1 = a + bi$, $\zeta_{-1} = c + di$, we find that

$$x_\nu = (a+c)\cos\frac{2\pi\nu}{5} + (-b+d)\sin\frac{2\pi\nu}{5}$$

$$y_\nu = (b+d)\cos\frac{2\pi\nu}{5} + (a-c)\sin\frac{2\pi\nu}{5},$$

and (2.9) is established. Replacing in the right sides ν by 2ν, we obtain (2.10).

So far we have only made general remarks on the f.F.s. of 5 terms which readily extend to the series for k terms. To obtain Theorem 1 we want to invert the real f.F.s.

(2.6), i.e. find the individual terms $\tilde{f}_\nu^1$ and $\tilde{f}_\nu^2$. This is where Douglas' idea comes in. From (2.3), with $\zeta_0 = 0$, and writing $\omega = \omega_1$, we obtain

$$z_{\nu+2} = (\zeta_1\omega_\nu\omega^2 + \zeta_{-1}\omega_\nu^{-1}\omega^{-2}) + (\zeta_2\omega_\nu^2\omega^{-1} + \zeta_{-2}\omega_\nu^{-2}\omega) ,$$

$$z_{\nu-2} = (\zeta_1\omega_\nu\omega^{-2} + \zeta_{-1}\omega_\nu^{-1}\omega^2) + (\zeta_2\omega_\nu^2\omega + \zeta_{-2}\omega_\nu^{-2}\omega^{-1}) ,$$

and therefore

$$z'_\nu = \frac{1}{2}(z_{\nu+2} + z_{\nu-2}) = \frac{1}{2}(\omega^2 + \omega^{-2})(\zeta_1\omega_\nu + \zeta_{-1}\omega_\nu^{-1}) + \frac{1}{2}(\omega + \omega^{-1})(\zeta_2\omega_\nu^2 + \zeta_{-2}\omega_\nu^{-2}).$$

But then, by (2.4), we have

$$z'_\nu = \tilde{f}_\nu^1 \cos\frac{4\pi}{5} + \tilde{f}_\nu^2 \cos\frac{2\pi}{5} . \tag{2.11}$$

Since $\cos(4\pi/5) = -\cos(\pi/5)$, all that we have to do now, is to invert the system of equations

$$\begin{aligned} z_\nu &= \tilde{f}_\nu^1 + \tilde{f}_\nu^2 \\ z'_\nu &= -\tilde{f}_\nu^1 \cos\frac{\pi}{5} + \tilde{f}_\nu^2 \cos\frac{2\pi}{5} . \end{aligned} \tag{2.12}$$

Since $\cos(\pi/5) = (1+\sqrt{5})/4$, $\cos(2\pi/5) = (-1+\sqrt{5})/4$, we readily find the solution of (2.12) to be given by

$$\begin{aligned} \tilde{f}_\nu^1 &= (-\frac{1}{\sqrt{5}} z_\nu + (1 + \frac{1}{\sqrt{5}})z'_\nu)\cdot \frac{1-\sqrt{5}}{2} \\ \tilde{f}_\nu^2 &= (\frac{1}{\sqrt{5}} z_\nu + (1 - \frac{1}{\sqrt{5}})z'_\nu) \cdot \frac{1+\sqrt{5}}{2}. \end{aligned} \tag{2.13}$$

Introducing the new points

$$\begin{aligned} f_\nu^1 &= -\frac{1}{\sqrt{5}} z_\nu + (1 + \frac{1}{\sqrt{5}}) z'_\nu , \\ f_\nu^2 &= \frac{1}{\sqrt{5}} z_\nu + (1 - \frac{1}{\sqrt{5}})z'_\nu , \end{aligned} \tag{2.14}$$

we obtain the f.F.s. (2.6) in the form

$$z_\nu = \frac{1-\sqrt{5}}{2} f_\nu^1 + \frac{1+\sqrt{5}}{2} f_\nu^2 . \tag{2.15}$$

Let us now establish Theorem 1 for the case where $\Pi \subset \mathbb{C}$. From the first relation (2.14) we find that

$$f_\nu^1 - z'_\nu = \frac{1}{\sqrt{5}} (z'_\nu - z_\nu) , \tag{2.16}$$

while the second relation (2.14) shows that

$$f_\nu^2 - z'_\nu = -\frac{1}{\sqrt{5}} (z'_\nu - z_\nu) . \tag{2.17}$$

(2.16), (2.17), are identical with the relations (1.3) that we wished to establish.

Why are the polgons Π^1 and Π^2, defined by (1.4) and (1.5), affine regular? From (2.13) and (2.14) we find that

$$f_\nu^1 = \tilde{f}_\nu^1 \Big/ \frac{1-\sqrt{5}}{2}, \qquad f_\nu^2 = \tilde{f}_\nu^2 \Big/ \frac{1+\sqrt{5}}{2}, \tag{2.18}$$

while we know by (2.7), (2.9), (2.10) that the polygons $\tilde{\Pi}^1$ and $\tilde{\Pi}^2$ are affine regular. A proof of Theorem 1, for the case where $\Pi \subset \mathbb{C}$, follows from the relations (2.18).

3. A proof of Theorem 1 if $\Pi \subset R^3$. We point out first that the definition of the pentagons (1.4) and (1.5), by the relations (1.2) and (1.3), remains valid in any real vector space, in particular for R^3. The only statements still in doubt are (2.9) and (2.10).

Let

$$F = (\Pi, \Pi^1, \Pi^2) \tag{3.1}$$

denote the space figure obtained by (1.2) and (1.3), and let

$$F_{xy} = (\Pi_{xy}, \Pi^1_{xy}, \Pi^2_{xy}), \quad F_{xz} = (\Pi_{xz}, \Pi^1_{xz}, \Pi^2_{xz}) \tag{3.2}$$

be its orthogonal projections onto the coordinate planes xOy and xOz, respectively, Since the construction of F is affine invariant, it is clear that we can apply to the plane figures (3.2) the results of the last section, in particular

(3.3) the pentagons Π^1_{xy} and Π^1_{xz} are affine regular.

We now appeal to the following most elementary

Lemma 1. *If the space pentagon*

$$\Pi^1 = (x_\nu, y_\nu, z_\nu) \qquad (\nu = 0,1,2,3,4) \tag{3.4}$$

has plane projections

$$\Pi^1_{xy} = (x_\nu, y_\nu), \quad \Pi^1_{xz} = (x_\nu, z_\nu) \tag{3.5}$$

which are affine regular pentagons, then Π^1 *itself is a plane pentagon which is affine regular.*

Proof: The affine regular pentagons (3.5) admit representations of the form

$$\left\{\begin{aligned} x_\nu &= a\cos\frac{2\pi\nu}{5} + b\sin\frac{2\pi\nu}{5}, \\ y_\nu &= c\cos\frac{2\pi\nu}{5} + d\sin\frac{2\pi\nu}{5}, \end{aligned}\right. \qquad \left\{\begin{aligned} x_\nu &= a'\cos\frac{2\pi\nu}{5} + b'\sin\frac{2\pi\nu}{5} \\ z_\nu &= e\cos\frac{2\pi\nu}{5} + f\sin\frac{2\pi\nu}{5}. \end{aligned}\right. \tag{3.6}$$

On comparing the first two equations of (3.6) we conclude that we must have $a = a'$, $b = b'$, and so

$$\begin{aligned} x_\nu &= a\cos\frac{2\pi\nu}{5} + b\sin\frac{2\pi\nu}{5}, \\ y_\nu &= c\cos\frac{2\pi\nu}{5} + d\sin\frac{2\pi\nu}{5}, \\ z_\nu &= e\cos\frac{2\pi\nu}{5} + f\sin\frac{2\pi\nu}{5}. \end{aligned} \tag{3.7}$$

It follows that Π^1 is an affine regular pentagon in the plane defined by the oblique coordinate system of the two vectors $u = (a,c,e)$ and $v = (b,d,f)$. This completes our proof of Theorem 1.

Remarks. 1. The two pentagons Π^1 and Π^2 of Theorem 1 lie in different planes, but have as common center the centroid O of the vertices of Π. The problem of choosing Π so as to maximize the artistic effect of the entire structure is not mathematical and is, of course, hopeless.

2. Douglas' fortunate idea is to construct the pentagons Π^1 and Π^2, and not the pentagons

(3.8) $$\tilde{\Pi}^1 = \frac{1-\sqrt{5}}{2}\,\Pi^1 , \quad \tilde{\Pi}^2 = \frac{1+\sqrt{5}}{2}\,\Pi^2$$

which provide the final harmonic analysis

(3.9) $$\Pi = \tilde{\Pi}^1 + \tilde{\Pi}^2$$

according to (2.8). This idea simplifies considerably the final construction, because finding the pentagons (3.8) themselves would require two homothetic images with center O, a cumbersome complications.

4. The harmonic analysis of a skew heptagon. Our application of the f.F.s. to Douglas' theorem readily suggests the way to generalize his result to closed skew polygon having k vertices. Having in mind further outdoor sculptures, we restrict our discussion to the case when $k = 7$, hence

$$\Pi = (z_0, z_1, \ldots, z_6) \tag{4.1}$$

is a heptagon. We have omitted the case when $k = 6$ for the reason that regular star-shaped hexagons are not particularly interesting. We commence our discussion by assuming that

$$\Pi \subset \mathbb{C}, \tag{4.2}$$

when the z_ν are complex numbers. Their f.F.s. and its inverse formulae are

$$z_\nu = \sum_{\alpha=0}^{6} \zeta_\alpha \omega_\nu^\alpha, \qquad \zeta_\nu = \frac{1}{7} \sum_{\alpha=0}^{6} z_\alpha \omega_\nu^{-\alpha} \qquad (\nu = 0, \ldots, 6) \tag{4.3}$$

where $\omega_\nu = \exp(2\pi\nu/7)$. Again we assume that $z_0 + z_1 + \ldots + z_6 = 0$, hence $\zeta_0 = 0$, and folding the f.F.s., as in (2.3), we obtain

$$z_\nu = (\zeta_1 \omega_\nu + \zeta_{-1} \omega_\nu^{-1}) + (\zeta_2 \omega_\nu^2 + \zeta_{-2} \omega_\nu^{-2}) + (\zeta_3 \omega_\nu^3 + \zeta_{-3} \omega^{-3}) . \tag{4.4}$$

The midpoint of the side of Π that is opposite to the vertex z_ν is

$$z'_\nu = \frac{1}{2} (z_{\nu+3} + z_{\nu-3}) . \tag{4.5}$$

However, now we also need the further midpoint

$$z'' = \frac{1}{2} (z_{\nu+2} + z_{\nu-2}) . \tag{4.6}$$

From (4.4), and writing $\omega_1 = \omega$, we obtain

$$z_{\nu\pm3} = (\zeta_1 \omega_\nu \omega^{\pm3} + \zeta_{-1} \omega_\nu^{-1} \omega^{\mp3}) + (\zeta_2 \omega_\nu^2 \omega^{\mp1} + \zeta_{-2} \omega_\nu^{-2} \omega^{\pm1}) + (\zeta_3 \omega_\nu^3 \omega^{\pm2} + \zeta_{-3} \omega_\nu^{-3} \omega^{\mp2}),$$

whence

$$z'_\nu = \frac{\omega^3 + \omega^{-3}}{2} \tilde{f}_\nu^1 + \frac{\omega + \omega^{-1}}{2} \tilde{f}_\nu^2 + \frac{\omega^2 + \omega^{-2}}{2} \tilde{f}_\nu^3 , \tag{4.7}$$

if we write

$$\tilde{f}_\nu^j = \zeta_j \omega_\nu^j + \zeta_{-j} \omega_\nu^{-j} \qquad (j = 1,2,3 \ ; \ \nu = 0, \ldots, 6). \tag{4.8}$$

Likewise we obtain from (4.4) that

$$z_{\nu\pm2} = (\zeta_1 \omega_\nu \omega^{\pm2} + \zeta_{-1} \omega_\nu^{-1} \omega^{\mp2}) + (\zeta_2 \omega_\nu^2 \omega^{\mp3} + \zeta_{-2} \omega_\nu^{-2} \omega^{\pm3}) + (\zeta_3 \omega_\nu^3 \omega^{\mp1} + \zeta_{-3} \omega_\nu^{-3} \omega^{\pm1}),$$

whence

$$z''_\nu = \frac{\omega^2 + \omega^{-2}}{2} \tilde{f}_\nu^1 + \frac{\omega^3 + \omega^{-3}}{2} \tilde{f}_\nu^2 + \frac{\omega + \omega^{-1}}{2} \tilde{f}_\nu^3 . \tag{4.9}$$

By (4.4) and (4.8) we see that the real f.F.s. of Π is

$$z_\nu = \tilde{f}^1_\nu + \tilde{f}^2_\nu + \tilde{f}^3_\nu \, . \tag{4.10}$$

As in the case of pentagons, the analogue of Douglas' theorem will arise if we invert the 3×3 system of equations (4.10), (4.7), (4.9). Writing

$$\Omega_j = \frac{1}{2}(\omega^j + \omega^{-j}) = \cos\frac{2\pi j}{7}, \qquad (j = 1,2,3). \tag{4.11}$$

We are to solve the system

$$\begin{aligned} z_\nu &= \tilde{f}^1_\nu + \tilde{f}^2_\nu + \tilde{f}^3_\nu \\ z'_\nu &= \Omega_3\tilde{f}^1_\nu + \Omega_1\tilde{f}^2_\nu + \Omega_2\tilde{f}^3_\nu \, , \\ z''_\nu &= \Omega_2\tilde{f}^1_\nu + \Omega_3\tilde{f}^2_\nu + \Omega_1\tilde{f}^3_\nu \, . \end{aligned} \tag{4.12}$$

In terms of the inverse matrix

$$\begin{Vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{Vmatrix} = \begin{Vmatrix} 1 & 1 & 1 \\ \Omega_3 & \Omega_1 & \Omega_2 \\ \Omega_2 & \Omega_3 & \Omega_1 \end{Vmatrix}^{-1} \tag{4.13}$$

the solutions are

$$\tilde{f}^j_\nu = A_j\, z_\nu + B_j\, z'_\nu + C_j\, z''_\nu \, , \qquad (j = 1,2,3). \tag{4.14}$$

By (4.8) it is clear that the three heptagons

$$\tilde{\Pi}^j = (\tilde{f}^j_0, \tilde{f}^j_1, \tilde{f}^j_2, \tilde{f}^j_3, \tilde{f}^j_4, \tilde{f}^j_5, \tilde{f}^j_6), \qquad (j = 1,2,3), \tag{4.15}$$

are affine images of the three regular heptagons

$$\begin{aligned} &(1,\omega,\omega^2,\omega^3,\omega^4,\omega^5,\omega^6),\ (1,\omega^2,\omega^4,\omega^6,\omega,\omega^3,\omega^5) \\ &(1,\omega^3,\omega^6,\omega^2,\omega^5,\omega,\omega^4), \end{aligned} \tag{4.16}$$

respectively. In terms of the heptagons (4.15) we may write (4.10) as

$$\Pi = \tilde{\Pi}^1 + \tilde{\Pi}^2 + \tilde{\Pi}^3 \, . \tag{4.17}$$

However, the heptagons (4.15) are <u>not</u> the ones that we wish to construct. Rather, following Douglas' lead, we introduce the weights

$$\alpha_j = \frac{A_j}{s_j}\, , \quad \beta_j = \frac{B_j}{s_j}\, , \quad \gamma_j = \frac{C_j}{s_j}\, , \quad \text{where } s_j = A_j + B_j + C_j \, , \tag{4.18}$$

and want to construct the heptagons,

$$\Pi^j = (f^j_0, f^j_1, f^j_2, f^j_3, f^j_4, f^j_5, f^j_6), \qquad (j = 1,2,3), \tag{4.19}$$

having vertices given by

(4.20) $$f_\nu^j = \alpha_j z_\nu + \beta_j z'_\nu + \gamma_j z''_\nu , \qquad (j = 1,2,3).$$

We state our results as

Theorem 2. Let

(4.21) $$\Pi = (z_0, z_1, \ldots, z_6)$$

be a skew heptagon in R^3, and let

(4.22) $$z'_\nu = \frac{1}{2}(z_{\nu+3} + z_{\nu-3}), \qquad z''_\nu = \frac{1}{2}(z_{\nu+2} + z_{\nu-2})$$

be the midpoints of appropriate sides and chords of Π. By (4.11), (4.13) and (4.18) we define the three sets of numerical weights

(4.23) $$\alpha_j, \beta_j, \gamma_j , \quad \alpha_j + \beta_j + \gamma_j = 1 , \qquad (j = 1,2,3).$$

In each of the seven triangles

(4.24) $$T_\nu = (z_\nu, z'_\nu, z''_\nu) \qquad (\nu = 0,\ldots,6)$$

we define the three points

(4.25) $$f_\nu^1, f_\nu^2, f_\nu^3$$

as the centroids of T_ν with the three sets of weights (4.23), respectively. Equivalently, (4.25) are defined by the equations (4.20). Then the three heptagons

(4.26) $$\Pi^j = (f_0^j, f_1^j, f_2^j, f_3^j, f_4^j, f_5^j, f_6^j), \qquad (j = 1,2,3),$$

are plane heptagons and they are affine images of the regular heptagons (4.16), respectively.

Our Theorem 2 is, of course, fully established if we assume that $\Pi \subset \mathbb{C}$. That it remains true if $\Pi \subset R^3$ follows from reasonings similar to those used in extending Theorem 1 from R^2 to R^3, in particular from the lemma: If a heptagon Π in R^3, has two affine regular plane projections, then Π itself is plane and affine regular.

5. <u>The construction of a space model illustrating Theorem 2.</u> By this we mean the construction of the figure

(5.1) $$F = (\Pi, \Pi^1, \Pi^2, \Pi^3),$$

where Π, Π^1, Π^2, Π^3, are the heptagons of Theorem 2. This could be done graphically on a sheet of paper by the methods of Descriptive Geometry. However, we have in mind a 3-dimensional structure made out of thin (wooden) sticks.

For this purpose we need the numerical values of the weights (4.18). With sufficient accuracy for any physical construction, these are as follows:

(5.2) $$\left\| \begin{matrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{matrix} \right\| = \left\| \begin{matrix} -.08627 & .69859 & .38768 \\ .78485 & 1.08626 & -.87111 \\ .30141 & .21515 & .48344 \end{matrix} \right\|$$

(5.3) $$s_1 = -1.24697, \quad s_2 = .44504, \quad s_3 = 1.80193 .$$

The construction of the 14 points z'_ν and z''_ν by the formulae (4.22) presents no difficulties. These also determine the 7 triangles (4.24).

In the plane of each T_ν we are now to construct the centroids (4.25) for the three sets of weights (4.23). Here we use the following lemma which is too elementary to require a proof (The reader is asked to supply a diagram).

<u>Lemma 2.</u> <u>Let</u>

(5.4) $$T = (z, z', z'')$$

<u>be a triangle, and let</u>

(5.5) $$f = \alpha z + \beta z' + \gamma z''$$

<u>be its centroid for the weights</u> α, β, γ, <u>with</u> $\alpha+\beta+\gamma = 1$.

<u>If</u> h <u>denotes the intersection of the line joining</u> z <u>to</u> z', <u>with the line joining</u> z'' <u>to</u> f, <u>then the relations</u>

(5.6) $$h-z' = \rho(z' - z), \quad f-h = \sigma(h - z'')$$

hold, where

(5.7) $$\rho = \frac{\alpha}{\alpha+\beta}, \qquad \sigma = -\gamma .$$

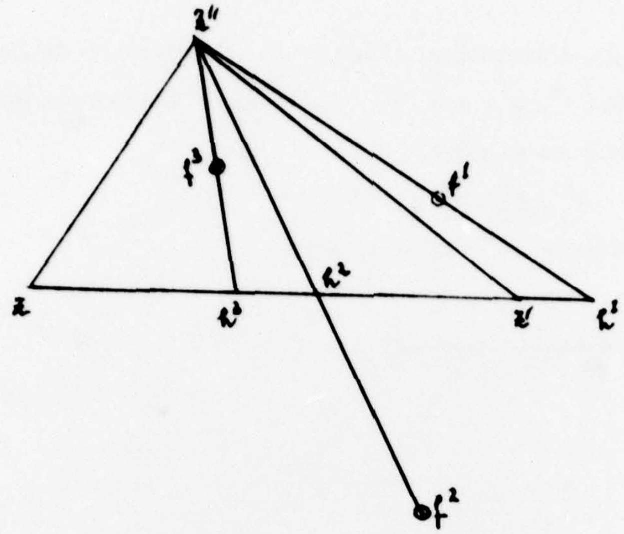

Figure 2

We apply Lemma 2 to each T_ν with the sets of weights (5.2). We drop the subscript ν and show in Figure 2 the location of the centroids f^1, f^2, f^3 in the plane of the triangle $T = (z,z',z")$. Using Lemma 2 and the numerical values (5.2), we obtain the rela-relations

$$
\begin{aligned}
& h^1 - z' = \rho_1(z' - z), \quad && f^1 - h^1 = \sigma_1(h^1 - z") \\
(5.8) \qquad & h^2 - z' = \rho_2 (z' - z), \quad && f^2 - h^2 = \sigma_2(h^2 - z") \\
& h^3 - z' = \rho_3(z' - z), \quad && f^3 - h^3 = \sigma_3(h^3 - z") .
\end{aligned}
$$

The numerical values of the ratios ρ_j and σ_j, given by (5.7) and (5.2), are

$$
\begin{aligned}
& \rho_1 = .14089 \qquad && \sigma_1 = -.38768 \\
(5.9) \qquad & \rho_2 = -.41946 \qquad && \sigma_2 = .87111 \\
& \rho_3 = -.58350 \qquad && \sigma_3 = -.48344 .
\end{aligned}
$$

The locations of the points h^j and f^j in Figure 2, are drawn to scale. For any other triangle $T_\nu = (z_\nu, z'_\nu, z"_\nu)$ the corresponding diagram is the image of Figure 2 by the affine transformation mapping T onto T_ν.

Our Figure 3 shows a 2-dimensional illustration of Theorem 2. It shows the three affine regular heptagons Π^1, Π^2, and Π^3. In order to simplify the drawing it shows only the construction of the three vertices

$$f_1^1 , f_1^2 , f_1^3 ,$$

corresponding to the triangle $T_1 = (z_1, z_1', z_1'')$.

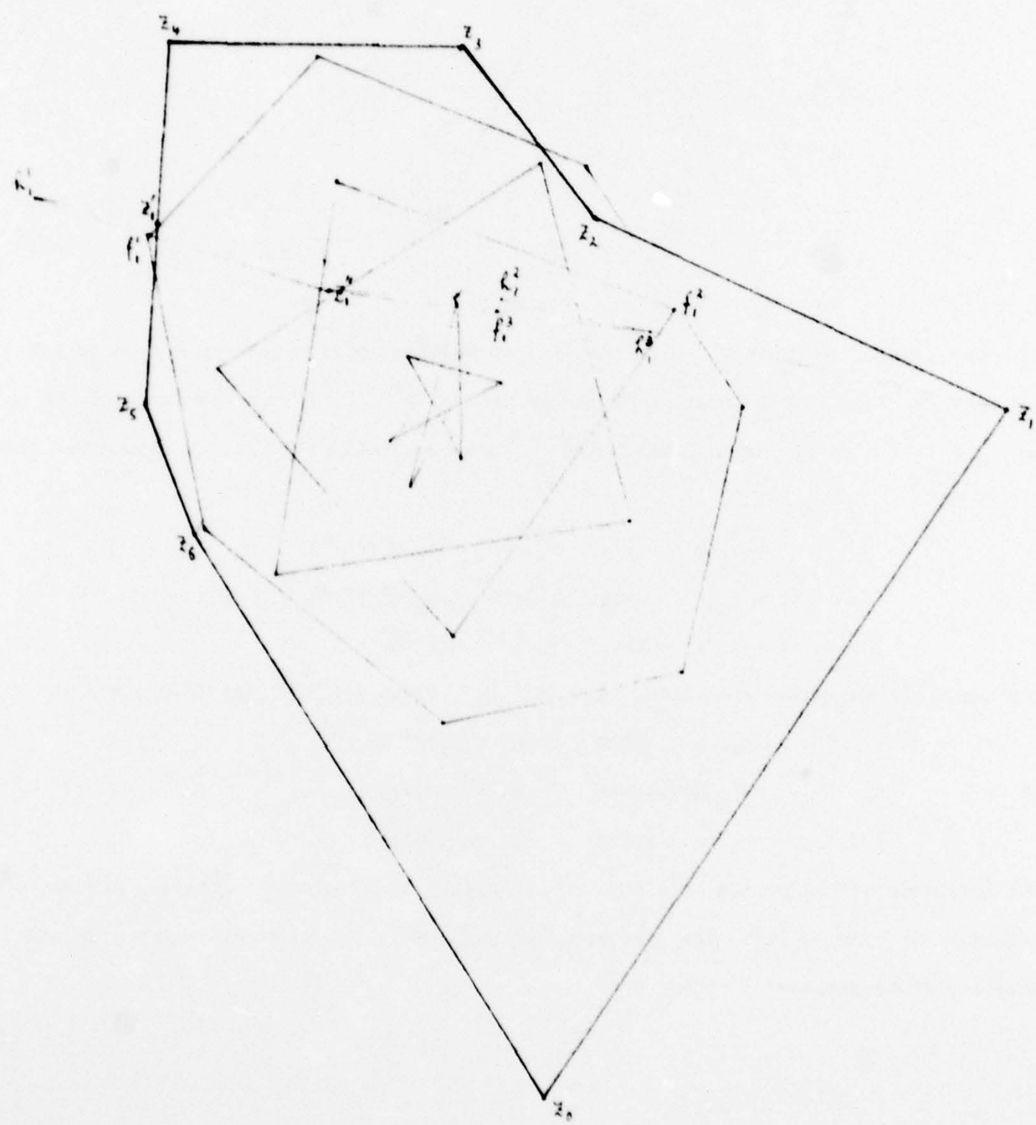

Figure 3

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In [1, 125] and again in [3] Jesse Douglas established the following

Theorem 1. Let

$$\Pi = (z_0, z_1, z_2, z_3, z_4), \qquad (z_{\nu+5} = z_\nu),$$

be a closed skew pentagon in R^3, viewed as a vector space. Let

$$z'_\nu = \frac{1}{2}(z_{\nu+2} + z_{\nu-2}) \qquad (\nu = 0,1,2,3,4)$$

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Abstract (continued)

be the midpoint of the side $[z_{\nu-2}, z_{\nu+2}]$ which is opposite to the vertex z_ν.

For each ν determine on the line joining z_ν to z'_ν, the points f^1_ν, f^2_ν, such that

$$f^1_\nu - z'_\nu = \frac{1}{\sqrt{5}}(z'_\nu - z_\nu), \quad f^2_\nu - z'_\nu = -\frac{1}{\sqrt{5}}(z'_\nu - z_\nu).$$

Then

$$\Pi^1 = (f^1_0, f^1_1, f^1_2, f^1_3, f^1_4)$$

is a plane and affine regular pentagon, and

$$\Pi^2 = (f^2_0, f^2_1, f^2_2, f^2_3, f^2_4)$$

is a plane and affine regular starshaped pentagon.

By an affine regular (starshaped) pentagon we mean an affine image of a regular (starshaped) pentagon.

It is shown here that the natural and inevitable source of Theorem 1 is the finite Fourier series of five terms. The affine regular pentagons Π^1 and Π^2 represent essentially the harmonic analysis of the pentagon Π. Placing the origin O of R^3 in the centroid of the vertices of Π, the complete harmonic analysis of Π is summarized by the relation

$$\Pi = \frac{1-\sqrt{5}}{2}\Pi^1 + \frac{1+\sqrt{5}}{2}\Pi^2 .$$

The Figure 1 shows a 2-dimensional illustration of Theorem 1, but this gives only a faint idea of the appearance of a 3-dimensional structure. The author made a 3-dimensional structure out of 20 thin wooden sticks, and was struck by its appropriateness as a source of outdoor sculptures.

Theorem 2 (§4) describes the analogue of Theorem 1 for skew heptagons in R^3. Figure 3, of §5, shows a 2-dimensional illustration of Theorem 2. A 3-dimensional model would be very desirable.